Some Properties of Equilateral Triangles

(With a brief introduction to techniques commonly used in mathematical proofs)

During the previous class, some of you asked me two important questions:

- Can we use geometry to derive formulas for areas and volumes of objects?
- How can we know whether a formula for, say, the area of a triangle, is true for every triangle of the same type?

The second question is especially good: it goes to the heart of mathematics, and of reality itself. A branch of philosophy called *epistemology* is dedicated to such questions. We don't have space here to give that question the attention that it deserves, but we can, with benefit, present some of the general ideas. Why not do so as we derive formulas for some properties of equilateral triangles?

Two philosophical doctrines: Realism and idealism

Realism says that there is a "real world" of objects that exist whether someone is there to observe them or not.

In contrast, **idealism** says, in effect, that the "world" is a creation of our minds.

Most scientists take realism for granted, without recognizing that a fundamental question is involved.

Regarding the assumptions that are common in mathematical proofs

Some students are surprised to learn that all mathematics is based upon assumptions, most of which are given impressive names like "postulates" or "axioms". The truth is that it's impossible to reason without making certain assumptions. At a minimum, one must assume one's own existence.

To derive formulas for areas and volumes of geometrical shapes, we must assume much more. First, we assume that these objects "exist" in some sense. (See note in the margin about realism and idealism.) The idea that characteristics such as length, area, and volume can be expressed by means of numbers is also based upon assumptions. So is the idea that these characteristics are related through mathematical operations. (I'll leave out the details.)

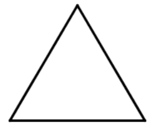
There's no denying that our everyday experiences show that arithmetic and geometry can predict, often with considerable precision, that which occurs in the world of physical objects. However, those experiences do not prove that the assumptions behind the mathematics are true.

When deriving formulas, we must be careful to list our assumptions, unless we are certain that they will be well known to our readers. Otherwise, we risk miscommunicating with them (and committing errors ourselves). To make communication easier, mathematicians give names to the assumptions that that are used most frequently. The two assumptions that interest us most in this document are known as *The Rule of Universal Specification* and *The Rule of Universal Generalization*:

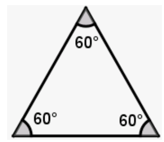
The Rule of Universal Specification
This rule is obvious, but we do need to state it. For our purposes, it says that if all objects of a certain type have a certain property, then each object of that type has that property. Here are two examples:

All humans are mortal. Fred is human. Therefore, Fred is mortal.

All equilateral triangles have three equal angles. The triangle shown below is equilateral.



Therefore, it has three equal angles.



The Rule of Universal Generalization
This rule is not easily stated. For our purposes, it says that if we can show that a formula must be true for any object chosen at random from a class, then the formula must be true for all objects that belong to that class. What does "chosen at random" mean? We can best answer that question by using this rule in a derivation.

Derivations of formulas for properties of equilateral triangles

When we begin to work on any problem, it's good to ask ourselves

What do we want?[1]

> **What do we want?**
> We want an equation that communicates the relationship be-tween an equilateral trian-gle's height and the length of its edges.

In our case, that's a good question. No one is making us investigate the properties of equilateral triangles; it's something that we decided to do for our own reasons. We can look at whatever properties we might wish, or that appear to be worth our time.

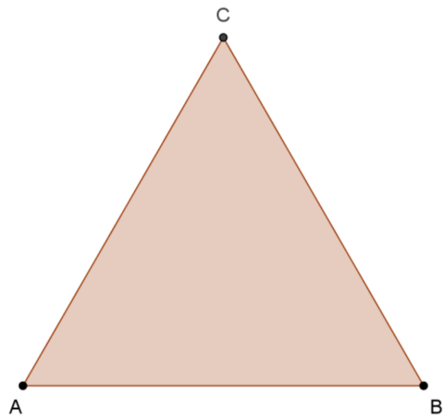
How about if we derive an equation that communicates the relationship between an equilateral triangle's height and the length of its edges? How should we get started?

A good piece of advice is

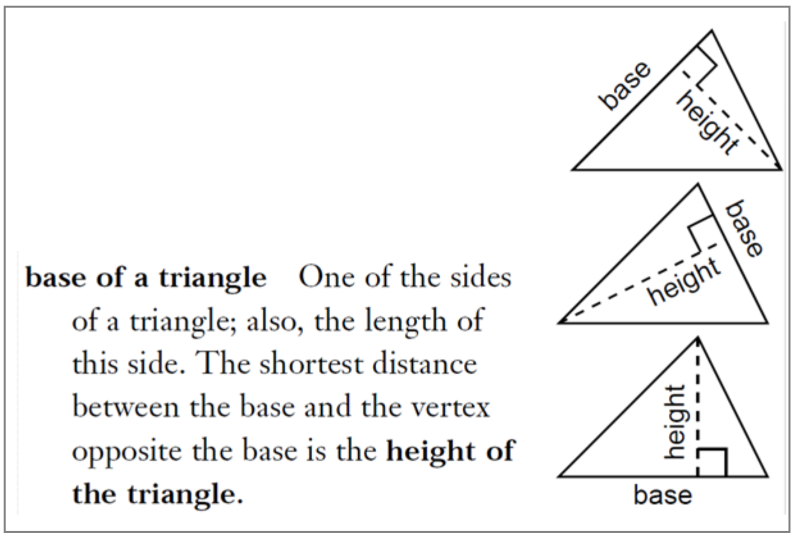
Any time you work on a problem in mathematics, draw a diagram that will help you see what's going on.[2]

So, let's draw an equilateral triangle.

[1] This suggestion comes from my favorite math book: *Thinking Mathematically* J. Mason, L. Burton, and, K. Stacey. It's available in Spanish as *Como razonar ma-tematicamente* from the publisher TRILLAS (ISBN-10: 607171544X, ISBN-13: 978-6071715449).
[2] A suggestion attributed to Alan Schoenfeld of the University of California, Berkeley



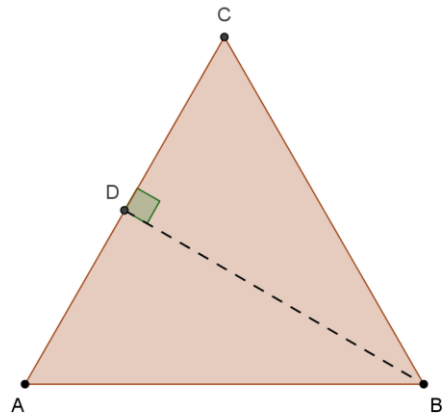
Now, what do we mean by the “height” of a triangle? A good definition is found in the *Everyday Mathematics*[3] glossary:



base of a triangle One of the sides of a triangle; also, the length of this side. The shortest distance between the base and the vertex opposite the base is the **height of the triangle.**

Because all three sides of an equilateral triangle are identical, the heights from all three sides are the same. To be a little different, why not use the one shown on the next page?

[3] No title given: available online at http://www.wrightgroup.com/download/em/emglossary.pdf (retrieved 6 March 2012). © 1998 Everyday Learning Corporation.



Now, let's go back now to our question,

What do we want?

Our answer was

> *We want an equation that communicates the relationship between an equilateral triangle's height and the length of its edges.*

Thanks to our diagram, we can make that answer more specific:

> *We want an equation that communicates the relationship between the length of segment $\overline{BD}$ and (for example) of $\overline{AB}$.*

To make things a little more convenient, we can follow another good piece of advice:

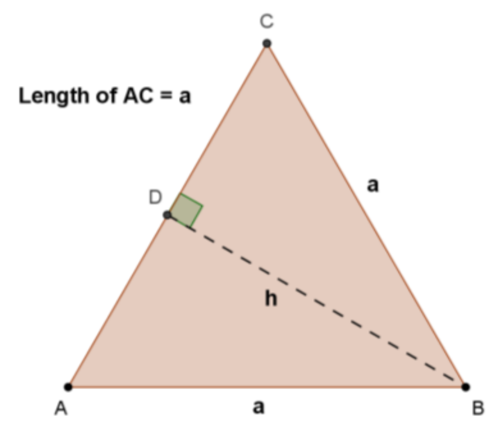
> *Ask yourself, "What can I introduce?"*

Sometimes, we might find it useful to "introduce" (or, "add") another line to a drawing. One of my own weaknesses in solving problems is that I don't do this often enough. Another thing that's often useful to "introduce" is a set of symbols to represent lengths, areas, or whatever other quantity might be involved in our problem. Let's do that now:

Define

a = Length of the triangle's sides. For example, of $\overline{AB}$.

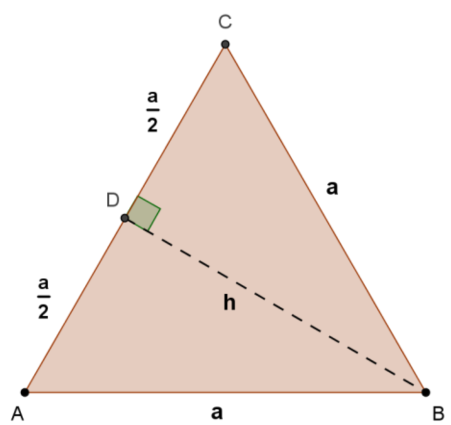
h = Length of $\overline{BD}$.



To continue, let's follow a third piece of good advice:

Ask yourself, "What do we know?"

One thing we know is that segment point *D* divides segment $\overline{AC}$ in half. Let's add that information to our diagram.



How do we know that point *D* divides $\overline{AC}$ in half, and that triangles *ADB* and *CDB* are right triangles?

We won't answer those questions here. They're good things to investigate outside of class.

We also know that Δ*ADB* and Δ*CDB* are right triangles. We know that the Pythagorean Theorem is true for all right triangles. To shorten this derivation, I'll just say that $\overline{AB}$ is the hypotenuse of Δ*ADB*, so

$$(BD)^2 + (AD)^2 = (AB)^2 \ . \qquad (1)$$

Now, we'll substitute *a* for *AB*, *h* for *BD*, and $\frac{a}{2}$ for *AD*. With these substitutions, Equation (1) becomes

$$(h)^2 + \left(\frac{a}{2}\right)^2 = (a)^2.$$

Let's stop here to notice that we've achieved our goal: we now have an equation that communicates the relationship between the height (*h*) of an equilateral triangle and the length of its sides (*a*). However, we should look for ways to make our equation simpler. For example, by combining the terms in which *a* appears:

$$(h)^2 + \left(\frac{a}{2}\right)^2 = (a)^2.$$

$$(h)^2 + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = (a)^2 - \left(\frac{a}{2}\right)^2$$

$$h^2 = a^2 - \frac{a^2}{4}$$

$$h^2 = \frac{3}{4}a^2 .$$

This result isn't bad, but we should probably simplify it even more by taking the square root of both sides:

$$\sqrt{h^2} = \sqrt{\frac{3}{4}a^2}$$

$h = a\frac{\sqrt{3}}{2}$, or equivalently, $h = \frac{a\sqrt{3}}{2}$.

We now have the sort of equation that we wanted, and in a convenient form. To check our work, we should measure *h* and *a* in our drawing, calculate the value of $a\frac{\sqrt{3}}{2}$, and see whether it's equal to *h*.

Always look for ways to check the formulas that you derive.

We should also try to see whether the formula makes sense. For example, what happens to *h* when the triangle gets bigger? That is, if *a* gets bigger, what should happen to *h*? We know that the bigger the triangle, the greater its height. Is that what the formula predicts? On the other hand, what if the triangle shrinks until it becomes a point? In that case, the height becomes zero. Does the formula predict that result?

Now, finally, we show where we used

The Rule of Universal Generalization and **The Rule of Universal Generalization**.

I promised that we'd see how to use The Rule of Universal Generalization and The Rule of Universal Generalization in this derivation, so you're probably wondering what happened to them. The truth is that we used those rules without saying so. We really should not have done that, so let's present an improved derivation, pointing out where those rules are used. All we need to do is add a few statements to the work we've already done. In the table that follows, comments ***written in blue italics*** are explanatory: they're not part of the derivation.

Improved derivation	Comments on each statement in our improved derivation
Let T, arbitrary, be an equilateral triangle.	***This is just a fancy way of saying, "Imagine any old equilateral triangle, and call it T."***
Let a be the length of T's sides.	***Here, we use <u>The Rule of Universal Specification</u> several times. As is usually the case, we don't* say *that we're using it; we just do it.*** ***All equilateral triangles share certain properties, and since <u>T</u> is an equilateral triangle, it has all of them.*** ***One of the properties that let us make the statement at the left is that the sides of an equilateral triangle are segments of equal length.*** ***All line segments, in turn, share certain important properties. Therefore, the segments that form <u>T</u> have those properties. One of those properties is that the length of a segment can be expressed as a non-negative number.***[4] **<u>IMPORTANT:</u>** ***We make no other assumptions about <u>T</u> or <u>a</u>. Especially, we don't assume a value for <u>a</u>.***
Now we make all of the statements that led us to our equation $h = a\frac{\sqrt{3}}{2}$.	***Again, we use <u>The Rule of Universal Specification</u> without saying so.*** ***Like all non-negative numbers, we can multiply <u>a</u>, divide it, etc.***
Because T was an arbitrary equilateral triangle, we can conclude that our formula is correct for all equilateral triangles. (End of derivation.)	***We are allowed to make this statement by <u>The Rule of Universal Generalization</u>. In deriving our equation, we assumed only that <u>T</u> had those properties that are common to all equilateral triangles. In other words, the assumption that <u>T</u> has only those properties leads to the conclusion that*** $\boldsymbol{h = a\frac{\sqrt{3}}{2}}$. ***Therefore, we may conclude that this equation is true for <u>all</u> equilateral triangles.***

For your convenience, the "Universal Rules" are presented again here:

<u>The Rule of Universal Specification</u>

If every object of a certain type has a certain property, then each object of that type has that property.

<u>The Rule of Universal Generalization</u>

If we can show that a formula is true for an object chosen at random from a class, then the formula is true for all objects that belong to that class.

[4] Conversely, every number can be represented as a line segment of appropriate length. Therefore, given any number x, there exists (at least in our minds!) an equilateral triangle with sides of length x. Although these ideas are taught as though they were obvious, they are quite subtle, and mathematicians did not manage to put them on solid footings until about 150 years ago.

Unfortunately, statements like the last one are usually left out of derivations and proofs, even in calculus-level textbooks. Most students never learn that we may make such conclusions, or what we must do in our derivations in order for us to employ The Rule of Universal Generalization.

I never learned of that Rule, or its use, until I studied techniques of formal proofs—at age 50! However, I believe that it can be taught successfully to elementary-school students.

We've now seen that we can indeed use geometry to derive equations (formulas) relating properties of figures. We've also seen

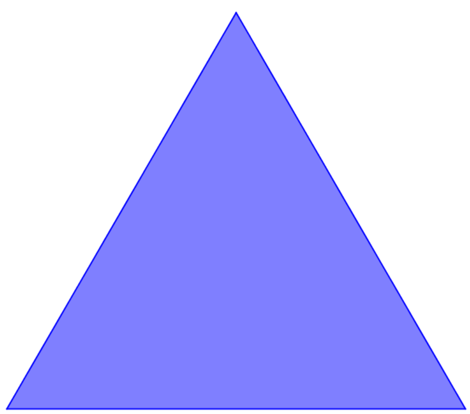
- How we "know" that those formulas are correct for all figures of the sort for which they were derived.
- That that "knowledge" is based upon an assumption called The Rule of Universal Generalization.

To tie up some loose ends, let's note that our derivation of the equation

$$h = a\frac{\sqrt{3}}{2}$$

was done correctly according to all the customs and rituals of that strange group of people called Mathematicians. Therefore, we can now add that relationship between height and length of sides to our list of properties common to all equilateral triangles.

Now, when we have to find the height of a given equilateral triangle, for example this one:



Length of each side: 5 meters

we can say, "<u>All</u> equilateral triangles have the property that the height is $\frac{\sqrt{3}}{2}$ times the length of the side, so that relation must be true for our triangle. Therefore, the height of our triangle is (5 meters) times $\frac{\sqrt{3}}{2}$, or about 4.33 meters. (Try it!)

One thing that's especially important to remember about this formula (and all others) is that it tells us a relationship between numbers. Specifically, it tells us that the number that expresses the height is equal to $\frac{\sqrt{3}}{2}$ times the number that expresses the length of the sides. Therefore, if we know the height of an equilateral triangle, and wish to know the length of its side, we can do so by solving

$$h = a\frac{\sqrt{3}}{2}$$

for *a*:

$$h = a\frac{\sqrt{3}}{2}$$

$$a\frac{\sqrt{3}}{2} = h$$

$$\frac{a\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{h}{\frac{\sqrt{3}}{2}}$$

$$a = h\left(\frac{2}{\sqrt{3}}\right)$$

$$a = \left(\frac{2\sqrt{3}}{3}\right)h\ .$$

Note that we did not have to "start from scratch" to derive this equation. That is, we didn't have to go back to our arbitrary triangle *T*. Having obtained the equation

$$h = a\frac{\sqrt{3}}{2}\ ,$$

and having said that *a* and *h* are non-negative numbers, we could transform that equation in any way we chose, as long as our transformations respected the properties of numbers and of equalities. In fact, that's the process by which we transformed the equation that we got directly from the Pythagorean Theorem,

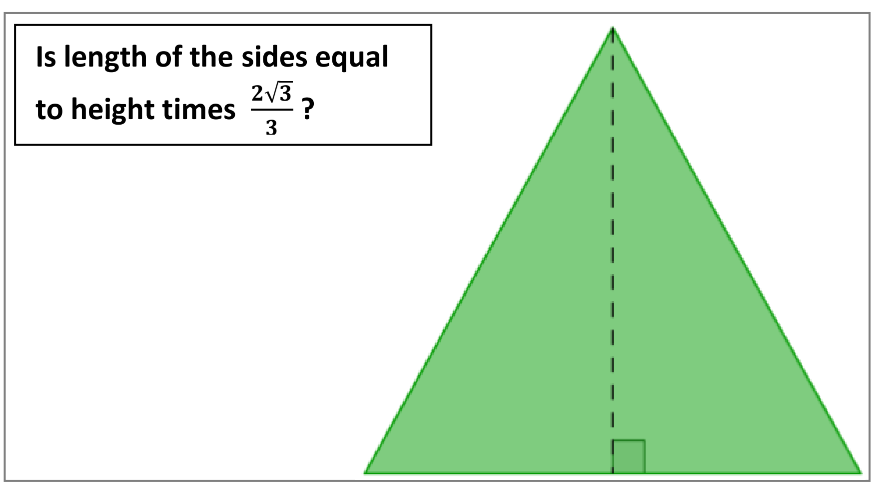
$$(h)^2 + \left(\frac{a}{2}\right)^2 = (a)^2.$$

into the more-useful form

$$h = a\frac{\sqrt{3}}{2}\ .$$

Of course, we should now test our formula on an equilateral triangle:

These transformations are purely "formal". That is, they just manipulate symbols according to the laws of algebra. Those rules are the same whether the formula deals with heights of triangles, or navigation of spacecraft. Why do those transformations give formulas that are correct in both cases? No one knows.



End